

Tiling a unit square with 8 squares

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Suppose a unit square is packed with n squares of side lengths s_1, s_2, \dots, s_n . We define $\psi_1(n) = \max \sum_{i=1}^n s_i$, where the maximum is taken over all possible packings of the unit square. Not a lot is known about the function ψ_1 . Erdős [1] asked whether $\psi_1(k^2 + 1) = k$. More generally, Erdős and Soifer [2] presented explicit packings that provided lower bounds of $\psi_1(n)$ for all (nonsquare) n ; they mentioned that these lower bounds appear to be good. Thus we have tentative values for $\psi_1(n)$.

In [3] Staton and Tyler introduced two modifications of ψ_1 as follows. Define a *right packing* to be a packing by squares whose sides are parallel to the sides of the unit square. Then $\psi_2(n)$ is defined to be $\max \sum s_i$ where the maximum is taken over all right packings with n squares. Also, $\psi_3(n)$, for $n \neq 2, 3, 5$, is defined to be $\max \sum s_i$, where the maximum is now taken over all right *tilings* with n squares. (A tiling is a packing where the unit square is completely filled. The unit square can be tiled with n squares for all values of n except for $n = 2, 3, 5$, thus the restriction on n in the definition of ψ_3 .) It is clear that $\psi_1(n) \geq \psi_2(n) \geq \psi_3(n)$. Staton and Tyler asked for what values of n we have $\psi_1(n) = \psi_2(n) = \psi_3(n)$.

There are some reasons to suspect that the three functions might be identical. The packings constructed by Erdős and Soifer in [2] are actually tilings, except when n differs by 1 from a square integer. Staton and Tyler in [3] took care of the case when n is one more than a perfect square by constructing tilings whose sums of edge lengths are the same as the Erdős-Soifer lower bounds. Thus if the Erdős-Soifer conjecture is correct, then $\psi_1(n) = \psi_2(n) = \psi_3(n)$ for all values of n except possibly when n is one less than a perfect square. In this note we show that, alas, $\psi_2(n) \neq \psi_3(n)$ when $n = 8$; more precisely, we show that $\psi_3(8) = 2.6$.

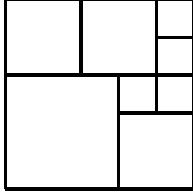
We first define our terminology and notation. All packings and tilings in this paper of the unit square, so we will omit the phrase “of the unit square” in what follows. If A is a square, its side length is denoted by s_A . If $\mathcal{C} = \{A_1, \dots, A_n\}$ is a collection of squares, we write $\sigma(\mathcal{C}) = \sigma(A_1, \dots, A_n)$ for $\sum s_{A_i}$.

Here is an upper bound due to Erdős; the proof below appeared in Erdős and Soifer [2].

Lemma 1. *If \mathcal{C} is a collection of n squares with total area A , then $\sigma(\mathcal{C}) \leq \sqrt{nA}$, with equality only if the n squares are the same size.*

Proof. Let s_1, \dots, s_n be the side lengths of the n squares. Apply the Cauchy-Schwarz inequality to the n -component vectors $(1, 1, \dots, 1)$ and (s_1, \dots, s_n) . \square

As an immediate consequence, we see that $\psi_3(8) \leq \sqrt{8}$. We also get a lower bound from an explicit construction: the tiling



shows that $\psi_3(8) \geq 2.6$. To show that $\psi_3(8) = 2.6$, we need to investigate the actual tiling in more detail.

Put our unit square so its corners are at $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Let \mathcal{C} be any tiling of this square with 8 tiles. For any c where $0 < c < 1$, we define \mathcal{C}_c to be the set of tiles whose interior intersect the vertical line $x = c$. We want to avoid the case where there is a tile with a vertical edge on the line $x = c$ (such a line is called *ambiguous* by Staton and Tyler [2]), so we will assume forthwith that the vertical line $x = c$ is not ambiguous. Thus $\sigma(\mathcal{C}_c) = 1$. Note that there is an unambiguous line as close as we want to an ambiguous line.

The values $c = 0$ and $c = 1$ are special. We call the line $x = 0$ the *left coast* and the line $x = 1$ the *right coast*. The *left coastal tiles* \mathcal{C}_0 are the tiles that have an edge on the left coast. Similarly, the *right coastal tiles* \mathcal{C}_1 are those tiles with an edge on the right coast. Their union is the set of *coastal tiles*. Tiles that are not coastal tiles are called *inland tiles*. There are not too many of these.

Lemma 2. *The sum of the side lengths of all inland tiles is less than 1.*

Proof. For any tiling \mathcal{C} , we know that $\sigma(\mathcal{C}) \leq \sqrt{8} < 3$. We have $\sigma(\mathcal{C}_0) = \sigma(\mathcal{C}_1) = 1$. If the sum of the side lengths of inland tiles is 1 or more, then $\sigma(\mathcal{C}) \geq 1 + 1 + 1 = 3$, a contradiction. \square

Lemma 3. *For any $0 < c < 1$, the set \mathcal{C}_c contains at least one coastal tile.*

Proof. Otherwise \mathcal{C}_c contains only inland tiles. Since $\sigma(\mathcal{C}_c) = 1$, this contradicts Lemma 2. \square

Lemma 4. *There is a tile $A \in \mathcal{C}_0$ and $B \in \mathcal{C}_1$ such that $s_A + s_B = 1$.*

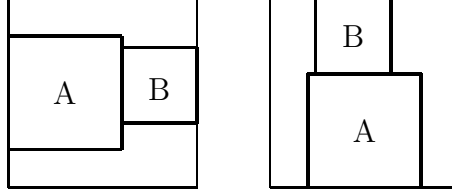
Proof. Let a denote the maximum side lengths of all left coastal tiles; similarly, let b denote the maximum side length of all right coastal tiles. If $a + b < 1$, then there exists a value x_0 (where $a < x_0 < 1 - b$) such that the line $x = x_0$ does not intersect any coastal tiles. This contradicts Lemma 3. Thus $a + b = 1$, which is what we want. \square

Note: the proof works just as well when we turn the tiling 90 degrees. Thus there exist two tiles, one with an edge on the line $y = 0$, and one with an edge on the line $y = 1$, such that the total edge lengths of these two tiles is 1.

Suppose as in Lemma 4 we have tiles $A \in \mathcal{C}_0$ and $B \in \mathcal{C}_1$ with $s_A + s_B = 1$.

Lemma 5. *One (or both) of A and B is a corner tile.*

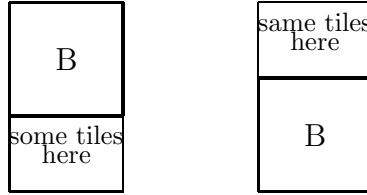
Proof. Suppose not.



Then rotating the tiling 90 degrees produces two inland tiles whose side lengths add up to 1, contradicting Lemma 2. \square

From now on we will assume, without loss of generality, that the left coastal tile A is a corner tile, with a corner at $(0, 0)$.

Note that we can further assume, without loss of generality, that B is also a corner tile, with a corner at $(1, 0)$. For any tiling where B has a corner at $(1, b)$, with $b > 0$, there is a similar tiling, with the same total edge length, where B has a corner at $(1, 0)$.



Clearly it does no harm to assume that $s_A \geq s_B$. (Simply reflect the tiling across the line $x = 1/2$ if necessary.) Thus our tiling contains a big tile A , with a corner at $(0, 0)$, where $s_A \geq 1/2$. There is also a tile B , with a corner at $(1, 0)$, where $s_B = 1 - s_A$. Similarly (see the note after Lemma 4) there is tile B' , with $s_{B'} = 1 - s_A$, which we can assume has a corner at $(0, 1)$. This is enough to show that $\psi_3(8)$ is not equal to $\psi_2(8)$.

Theorem 6. $\psi_2(8) > \psi_3(8)$.

Proof. In the standard 3×3 tiling, remove one tile. We then have a packing with 8 squares with total edge length $\frac{8}{3}$. Thus $\psi_2(8) \geq \frac{8}{3}$, so all we need to show is that $\psi_3(8) < \frac{8}{3}$.

Let $t = s_B$. The three tiles A , B , B' have total area $2t^2 + (1 - t)^2 = 1 - 2t + 3t^2$, leaving an area of $2t - 3t^2$ for the remaining 5 tiles. By Lemma 1,

the total side lengths of these 5 tiles is at most $\sqrt{5(2t - 3t^2)} = \sqrt{10t - 15t^2}$. Thus the total side lengths of all 8 tiles is at most $1 + t + \sqrt{10t - 15t^2}$. It is straightforward to verify that this function has a maximum at $t = 5/12$, with a maximum value of $8/3$. Thus we get that $\psi_3(8) \leq 8/3$.

Equality is achieved only if $t = 5/12$ and the 5 tiles are all the same size. Let us figure out what this size is. The 5 tiles have a total area of $2t - 3t^2 = 2 \cdot (5/12) - 3 \cdot (5/12)^2 = 45/144$, so each tile has area $9/144$, i.e., each tile has side length $3/12$. Now B (and B') must have an edge on the border of the unit square, so the remaining $7/12$ must be covered by tiles of side length $3/12$, i.e., an integer multiple of $3/12$ must be equal to $7/12$. This is impossible. Thus the optimal tiling must either have $t \neq 5/12$ or it must have 5 remaining tiles of different sizes. In either case, the total side length will be smaller than $8/3$. Hence $\psi_3(8) < 8/3$, as claimed. \square

We will now proceed with the proof that $\psi_3(8) = 2.6$. Suppose P is an optimal tiling, i.e., $\sigma(P)$ is maximal. We know that $\sigma(P) \geq 2.6$. As always, we assume without harm that P contains a corner tile A with a corner at $(0,0)$; there are also at least two tiles B and B' with edge lengths $s_B = s_{B'} = 1 - s_A$.

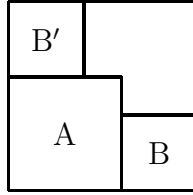
Lemma 7. *There are at most 3 tiles with edge lengths s_B .*

Proof. Suppose there are 4 tiles with edge lengths $t = s_B$. Then these 4 tiles, together with A , have total area $4t^2 + (1 - t)^2$, leaving an area of $2t - 5t^2$ for the remaining 3 tiles. The edge lengths of these 3 tiles sum up to at most $\sqrt{3(2t - 5t^2)} = \sqrt{6t - 15t^2}$, so the total edge length of all 8 tiles is at most $4t + (1 - t) + \sqrt{6t - 15t^2} = 1 + 3t + \sqrt{6t - 15t^2}$. It is straightforward to calculate that this function has a maximum value of $\frac{8+2\sqrt{6}}{5} < 2.58$ (at $t = \frac{4+\sqrt{6}}{20}$). Since $\sigma(P) \geq 2.6$, any tiling with 4 tiles of edge length s_B cannot be optimal. The situation is even worse if the tiling has more than 4 tiles of edge length s_B . \square

Lemma 8. *In an optimal tiling, there are exactly 3 tiles with edge lengths s_B .*

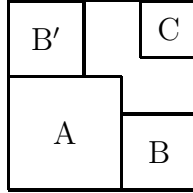
Proof. By Lemma 7 we need to show that there are at least 3 tiles with edge lengths s_B . We already know that there are 2 tiles, B and B' , with $s_{B'} = s_B$. Suppose there are no other tiles with edge length s_B ; we will derive a contradiction.

Recall that B can be assumed to be a right coastal tile with a corner at $(1, 0)$ and that B' can be assumed to have a corner at $(0, 1)$. Thus we have the following configuration.



If $s_A = s_B = 1/2$, then the remaining empty square of size $1/2$ -by- $1/2$ needs to be tiled by 5 squares. This is impossible. It follows that $s_A > 1/2$ (and so $s_B < 1/2$).

Let C denote the tile with a corner at $(1, 1)$.



There are 4 tiles that remain to be placed. At least 2 must share a border on the line $y = s_B$ with B (if there were only 1, then it must have edge length s_B); similarly, at least 2 must share a border on the line $x = s_B$ with B' . Thus there are exactly 2 tiles on top of B : one a right coastal tile (call it E) and one an inland tile, with a corner at (s_A, s_B) (call it D). Similarly, there are 2 tiles to the right of B' , one on the north border (call it E') and one with a corner at (s_B, s_A) (call it D').

Note that $s_E = s_{E'} = 1 - s_B - s_C$; also, $s_D = s_{D'} = s_B - s_E$. Thus the tiling is symmetric with respect to the main diagonal $y = x$.

Now consider the line connecting the northwest corner of A to the southeast corner of C . Since this is a diagonal line, it must intersect the interior of a tile, either D or E or D' or E' . But the symmetry of the tiling indicates that the aforementioned line must intersect the interior of *two* tiles, contradicting our requirement that the tiles don't overlap.

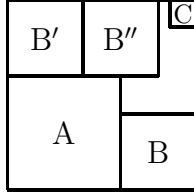
□

We note here a by-product of the proof: it cannot be the case that $s_A = s_B = 1/2$. Thus we have $s_A > s_B$.

Denote by B'' the third tile whose side length is equal to s_B . As above, we can assume that B'' lies adjacent to B' .

Theorem 9. $\psi_3(8) = \frac{13}{5}$.

Proof. Recall that C is the tile with a corner at $(1, 1)$. We have a configuration similar to the following.



Three tiles remain to be placed. Two of them are right coastal tiles and one—the one with a corner at (s_A, s_B) , which as before we call D —is an inland tile. Thus there are two inland tiles, B'' and D , and the total edge length of this tiling is $2 + s_B + s_D$.

If $s_D > \frac{1}{2}s_B$, then the two right coastal tiles besides B and C must each have edge length $s_B - s_D < \frac{1}{2}s_B$; thus $\sigma(\mathcal{C}_1) < s_B + \frac{1}{2}s_B + \frac{1}{2}s_B + s_C = 2s_B + s_C$. But looking at the north border we see that $2s_B + s_C \leq 1$, so $\sigma(\mathcal{C}_1) < 1$, a contradiction. Thus we must have $s_D \leq \frac{1}{2}s_B$, so the total length of the tiling is at most $2 + s_B + \frac{1}{2}s_B = 2 + \frac{3}{2}s_B$. Therefore $2 + \frac{3}{2}s_B \geq \frac{13}{5}$, i.e., $s_B \geq \frac{2}{5}$.

Now consider just the tiles A , B , B' , and B'' . Let $t = s_B$ as before. These tiles have total area $3t^2 + (1 - t)^2$, leaving an area of $2t - 4t^2$ to be covered with 4 tiles. By Lemma 1, the total edge lengths of these 4 tiles

is at most $\sqrt{4(2t - 4t^2)}$; thus the total edge length of all tiles is at most $3t + (1 - t) + \sqrt{4(2t - 4t^2)}$. For $t \geq \frac{2}{5}$, this function has a maximum value of $\frac{13}{5}$ (which occurs at $t = \frac{2}{5}$); thus $\sigma(C) \leq \frac{13}{5}$, as required.

□

Note: I do not know the value of $\psi_3(k^2 - 1)$ for $k > 3$. It is possible to show that $\psi_3(k^2 - 1) \geq k - \frac{1}{k-1}$ as follows. Start with a standard $(k+1) \times (k+1)$ tiling, and replace a $k \times k$ subsquare with a standard $(k-1) \times (k-1)$ tiling. We now have a tiling with $(k+1)^2 - k^2 + (k-1)^2 = k^2 + 2$ tiles. There are (a) $2k+1$ tiles with edge length $1/(k+1)$, and (b) $(k-1)^2$ tiles with edge length $\frac{k}{k^2-1}$. Pick any 2×2 subsquare in (b) and replace it with one big square; we now have a tiling with $k^2 - 1$ tiles. The total edge length of this tiling is $\frac{2k+1}{k+1} + \frac{k(k-1)^2}{k^2-1} - \frac{2k}{k^2-1} = k - \frac{1}{k-1}$.

References

- [1] P. Erdős, Some of my favorite problems in number theory. Comb. Week. Resenhas 2 (1995), no. 2, 165–186.
- [2] P. Erdős and A. Soifer, Squares in a Square. Geombinatorics, vol. IV, issue 4 (1995), 110–114.
- [3] W. Staton and B. Tyler, On the Erdős Square-Packing Conjecture. Geombinatorics, vol. XVII, issue 2 (2007), 88–94.